

# LOOPS IN $SU(2)$ AND FACTORIZATION

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ABSTRACT. We discuss a refinement of triangular factorization for the loop group of  $SU(2)$ .

## 0. INTRODUCTION

This paper is a sequel to [4]. The main purpose of the paper is to prove functional analytic generalizations of Theorems 0.1 and 0.2 below.

Let  $L_{fin}SU(2)$  ( $L_{fin}SL(2, \mathbb{C})$ ) denote the group consisting of functions  $S^1 \rightarrow SU(2)$  ( $SL(2, \mathbb{C})$ , respectively) having finite Fourier series, with pointwise multiplication. For example, for  $\zeta \in \mathbb{C}$  and  $n \in \mathbb{Z}$ , the function

$$S^1 \rightarrow SU(2) : z \rightarrow a(\zeta) \begin{pmatrix} 1 & \zeta z^{-n} \\ -\bar{\zeta} z^n & 1 \end{pmatrix},$$

where  $a(\zeta) = (1 + |\zeta|^2)^{-1/2}$ , is in  $L_{fin}SU(2)$ . It is known that  $L_{fin}SU(2)$  is dense in  $C^\infty(S^1, SU(2))$  (Proposition 3.5.3 of [6]). Also, if  $f(z) = \sum f_n z^n$ , let  $f^* = \sum \bar{f}_n z^{-n}$ . If  $f \in H^0(\Delta)$ , then  $f^* \in H^0(\Delta^*)$ , where  $\Delta$  is the open unit disk, and  $\Delta^*$  is the open unit disk at  $\infty$ .

**Theorem 0.1.** *Suppose that  $k_1 \in L_{fin}SU(2)$ . The following are equivalent:*

(a<sub>1</sub>)  $k_1$  is of the form

$$k_1(z) = \begin{pmatrix} a(z) & b(z) \\ -b^* & a^* \end{pmatrix}, \quad z \in S^1,$$

where  $a$  and  $b$  are polynomials in  $z$ , and  $a(0) > 0$ .

(b<sub>1</sub>)  $k_1$  has a factorization of the form

$$k_1(z) = a(\eta_n) \begin{pmatrix} 1 & -\bar{\eta}_n z^n \\ \eta_n z^{-n} & 1 \end{pmatrix} \dots a(\eta_0) \begin{pmatrix} 1 & -\bar{\eta}_0 \\ \eta_0 & 1 \end{pmatrix},$$

for some  $\eta_j \in \mathbb{C}$ .

(c<sub>1</sub>)  $k_1$  has triangular factorization of the form

$$\begin{pmatrix} 1 & 0 \\ \sum_{j=0}^n \bar{y}_j z^{-j} & 1 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix} \begin{pmatrix} \alpha_1(z) & \beta_1(z) \\ \gamma_1(z) & \delta_1(z) \end{pmatrix},$$

where  $a_1 > 0$  and the third factor is a polynomial in  $z$  which is unipotent upper triangular at  $z = 0$ .

Similarly, the following are equivalent:

(a<sub>2</sub>)  $k_2$  is of the form

$$k_2(z) = \begin{pmatrix} d^* & -c^* \\ c(z) & d(z) \end{pmatrix}, \quad z \in S^1,$$

where  $c$  and  $d$  are polynomials in  $z$ ,  $c(0) = 0$ , and  $d(0) > 0$ .

(b<sub>2</sub>)  $k_2$  has a factorization of the form

$$k_2(z) = a(\zeta_n) \begin{pmatrix} 1 & \zeta_n z^{-n} \\ -\bar{\zeta}_n z^n & 1 \end{pmatrix} .. a(\zeta_1) \begin{pmatrix} 1 & \zeta_1 z^{-1} \\ -\bar{\zeta}_1 z & 1 \end{pmatrix},$$

for some  $\zeta_j \in \mathbb{C}$ .

(c<sub>2</sub>)  $k_2$  has triangular factorization of the form

$$\begin{pmatrix} 1 & \sum_{j=1}^n \bar{x}_j z^{-j} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 & 0 \\ 0 & a_2^{-1} \end{pmatrix} \begin{pmatrix} \alpha_2(z) & \beta_2(z) \\ \gamma_2(z) & \delta_2(z) \end{pmatrix},$$

where  $a_2 > 0$  and the third factor is a polynomial in  $z$  which is unipotent upper triangular at  $z = 0$ .

*Remark.* The two sets of conditions are equivalent; they are intertwined by the outer involution  $\sigma$  of  $LSL(2, \mathbb{C})$  given by

$$(0.1) \quad \sigma \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} d & cz^{-1} \\ bz & a \end{pmatrix}.$$

This Theorem basically follows from results in [4], but it is possible to give a direct argument (not involving Lie theory). We will present this, and functional analytic generalizations, in Section 2.

The terminology regarding triangular factorization in the following theorem is reviewed in Section 1.

**Theorem 0.2.** (a) If  $\{\eta_i\}$  and  $\{\zeta_j\}$  are rapidly decreasing sequences of complex numbers, then the limits

$$k_1(z) = \lim_{n \rightarrow \infty} a(\eta_n) \begin{pmatrix} 1 & -\bar{\eta}_n z^n \\ \eta_n z^{-n} & 1 \end{pmatrix} .. a(\eta_0) \begin{pmatrix} 1 & -\bar{\eta}_0 \\ \eta_0 & 1 \end{pmatrix}$$

and

$$k_2(z) = \lim_{n \rightarrow \infty} a(\zeta_n) \begin{pmatrix} 1 & \zeta_n z^{-n} \\ -\bar{\zeta}_n z^n & 1 \end{pmatrix} .. a(\zeta_1) \begin{pmatrix} 1 & \zeta_1 z^{-1} \\ -\bar{\zeta}_1 z & 1 \end{pmatrix},$$

exist in  $C^\infty(S^1, SU(2))$ .

(b) Suppose  $g \in C^\infty(S^1, SU(2))$ . The following are equivalent:

(i)  $g$  has a triangular factorization  $g = lmau$ , where  $l$  and  $u$  have  $C^\infty$  boundary values.

(ii)  $g$  has a factorization of the form

$$g(z) = k_1(z)^* \begin{pmatrix} e^\chi & 0 \\ 0 & e^{-\chi} \end{pmatrix} k_2(z),$$

where  $\chi \in C^\infty(S^1, i\mathbb{R})$ , and  $k_1$  and  $k_2$  are as in (a).

(iii) The Toeplitz operator  $A(g)$  and the shifted Toeplitz operator  $A_1(g)$  are invertible.

**Remarks.** (a) Suppose that  $g \in L_{fin}SU(2)$ . The  $l$  and  $u$  factors in (i) are also in  $L_{fin}SL(2, \mathbb{C})$ , but they are essentially never unitary on  $S^1$ . On the other hand the factors  $k_j$  in (ii) are unitary, but in general they are not in  $L_{fin}SU(2)$  [If  $k_1, k_2 \in L_{fin}SU(2)$ , then  $\chi$  must be constant. Since  $L_{fin}SU(2)$  is dense in  $C^\infty(S^1, SU(2))$ , the parameterization in (ii) implies that generically  $g$  will correspond to nonconstant  $\chi$ .].

(b) There is a generalization of this Theorem with  $U(2)$  in place of  $SU(2)$ , where one restricts to loops in the identity component. We will restrict our attention to  $SU(2)$ , to simplify the exposition.

The outline of the paper is the following. Section 1 is a review of standard facts about triangular factorization.

In Sections 2 and 3, we prove Theorems 0.1 and 0.2, respectively. In these two sections, the main point is to extend the equivalences above to other function spaces, especially the critical Sobolev space  $W^{1/2,L^2}$ ; see Theorems 2.3 and 3.2.

It seems possible that there are  $L^2$  generalizations of these theorems. This is briefly discussed in Section 4. In the Appendix we discuss the combinatorial relation between  $x^*$  and  $\zeta$  in Theorem 0.1. This relation is central to the  $L^2$  question, and applications. Unfortunately this relation remains mysterious to me.

The generalization of the algebraic aspects of this paper from  $SU(2)$  to general simply connected compact groups is known ([4],[5]), but considerably more complicated. For  $SU(2)$  it suffices to consider one representation, the defining representation, which greatly simplifies everything.

**Notation.** Sobolev spaces will be denoted by  $W^s$ , and will always be understood in the  $L^2$  sense. The space of sequences satisfying  $\sum n|\zeta_n|^2 < \infty$  will be denoted by  $w^{1/2}$ . We will write  $Meas(S^1, SU(2))$  (rather than  $L^\infty(S^1, SU(2))$ ) for the group of (equivalence classes of) measurable maps. This group is usually equipped with the topology of convergence in measure, but this will not play a role in this paper.

We will use [3] as a general reference for Hankel and Toeplitz operators.

### 1. TRIANGULAR FACTORIZATION FOR $LSL(2, \mathbb{C})$

Suppose that  $g \in L^1(S^1, SL(2, \mathbb{C}))$ . A triangular factorization of  $g$  is a factorization of the form

$$(1.1) \quad g = l(g)m(g)a(g)u(g),$$

where

$$l = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} \in H^0(\Delta^*, SL(2, \mathbb{C})), \quad l(\infty) = \begin{pmatrix} 1 & 0 \\ l_{21}(\infty) & 1 \end{pmatrix},$$

$l$  has a  $L^2$  radial limit,  $m = \begin{pmatrix} m_0 & 0 \\ 0 & m_0^{-1} \end{pmatrix}$ ,  $m_0 \in S^1$ ,  $a(g) = \begin{pmatrix} a_0 & 0 \\ 0 & a_0^{-1} \end{pmatrix}$ ,  $a_0 > 0$ ,

$$u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in H^0(\Delta, SL(2, \mathbb{C})), \quad u(0) = \begin{pmatrix} 1 & u_{12}(0) \\ 0 & 1 \end{pmatrix},$$

and  $u$  has a  $L^2$  radial limit. Note that (1.1) is an equality of measurable functions on  $S^1$ . A Birkhoff (or Wiener-Hopf, or Riemann-Hilbert) factorization is a factorization of the form  $g = g_- g_0 g_+$ , where  $g_- \in H^0(\Delta^*, \infty; SL(2, \mathbb{C}), 1)$ ,  $g_0 \in SL(2, \mathbb{C})$ ,  $g_+ \in H^0(\Delta, 0; SL(2, \mathbb{C}), 1)$ , and  $g_\pm$  have  $L^2$  radial limits on  $S^1$ . Clearly  $g$  has a triangular factorization if and only if  $g$  has a Birkhoff factorization and  $g_0$  has a triangular factorization, in the usual sense of matrices.

**Proposition 1.** *Birkhoff and triangular factorizations are unique.*

*Proof.* If  $g_- g_0 g_+ = h_- h_0 h_+$  are two Birkhoff factorizations, then the function  $F$  equal to  $h_-^{-1} g_-$  for  $|z| \geq 1$  and  $(h_0 h_+)^{-1} g_0 g_+$  for  $|z| \leq 1$  is holomorphic on  $\mathbb{C} \setminus S^1$  and integrable on  $S^1$ . Integrability implies that the singularities along  $S^1$  are removable. Therefore  $F$  is constant, and the normalization conditions force  $F = 1$ . This implies uniqueness.  $\square$

*Remark.* In the definition of Birkhoff factorization, if the  $L^2$  condition is replaced by the weaker condition that  $g_{\pm}$  have pointwise radial limits *a.e.* on  $S^1$ , then factorization is not unique. For example

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{z+1}{z-1} & 0 \\ 0 & \frac{z-1}{z+1} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -\frac{z-1}{z+1} & 0 \\ 0 & -\frac{z+1}{z-1} \end{pmatrix}$$

is a factorization in this weaker sense. At least for the purposes of this paper,  $L^2$  appears to be the natural regularity condition in the definitions of factorization.

As in [6], consider the polarized Hilbert space

$$\mathcal{H} := L^2(S^1, C^2) = \mathcal{H}^+ \oplus \mathcal{H}^-,$$

where  $\mathcal{H}^+ = P_+ \mathcal{H}$  consists of  $L^2$ -boundary values of functions holomorphic in  $\Delta$ . If  $g \in L^\infty(S^1, SL(2, \mathbb{C}))$ , we write the bounded multiplication operator defined by  $g$  on  $\mathcal{H}$  as

$$M_g = \begin{pmatrix} A(g) & B(g) \\ C(g) & D(g) \end{pmatrix}$$

where  $A(g) = P_+ M_g P_+$  is the (block) Toeplitz operator associated to  $g$  and so on.

If  $g$  has the Fourier expansion  $g = \sum g_n z^n$ ,  $g_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ , then relative to the basis for  $\mathcal{H}$ :

$$(1.2) \quad \dots \epsilon_1 z, \epsilon_2 z, \epsilon_1, \epsilon_2, \epsilon_1 z^{-1}, \epsilon_2 z^{-1}, \dots$$

the matrix of  $M_g$  is block periodic of the form

$$\begin{array}{cccccc|cccc} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ \dots & a_0 & b_0 & a_1 & b_1 & | & a_2 & b_2 & \dots & \\ \dots & c_0 & d_0 & c_1 & d_1 & | & c_2 & d_2 & \dots & \\ \dots & a_{-1} & b_{-1} & a_0 & b_0 & | & a_1 & b_1 & \dots & \\ \dots & c_{-1} & d_{-1} & c_0 & d_0 & | & c_1 & d_1 & \dots & \\ \hline & & & & & & & & & \\ \dots & a_{-2} & b_{-2} & a_{-1} & b_{-1} & | & a_0 & b_0 & \dots & \\ \dots & c_{-2} & d_{-2} & c_{-1} & d_{-1} & | & c_0 & d_0 & \dots & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \end{array}$$

From this matrix form, it is clear that, up to equivalence,  $M_g$  has just two types of “principal minors”, the matrix representing  $A(g)$ , and the matrix representing the shifted Toeplitz operator  $A_1(g)$ , the compression of  $M_g$  to the subspace spanned by  $\{\epsilon_i z^j : i = 1, 2, j > 0\} \cup \{\epsilon_1\}$ . Relative to the basis (1.2), the involution  $\sigma$  defined by (0.1) is equivalent to conjugation by the shift operator, i.e. the matrix of  $M_{\sigma(g)}$  is obtained from the matrix for  $M_g$  by shifting one unit along the diagonal (in either direction: the result is the same, because  $M_g$  commutes with  $M_z$ , the square of the shift operator). Consequently the shifted Toeplitz operator is equivalent to the operator  $A(\sigma(g))$ .

**Theorem 1.1.** *Suppose that  $g \in L^\infty(S^1, SL(2, \mathbb{C}))$ .*

(a) *If  $A(g)$  is invertible, then  $g$  has a Birkhoff factorization, where*

$$(1.3) \quad (g_0 g_+)^{-1} = [A(g)^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, A(g)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}].$$

(b) *If  $A(g)$  and  $A_1(g)$  are invertible, then  $g$  has a triangular factorization.*

*Proof.* For part (a), let  $M$  denote the  $2 \times 2$  matrix valued loop on the right hand side of (1.3). The columns of this matrix are in  $\mathcal{H}^+$ . We must check that  $\det(M) = 1$  on  $\Delta$ . Because the entries of  $M$  are in  $L^2(S^1)$ ,  $\det(M) \in L^1(S^1)$ . Because  $\det(g) = 1$  on  $S^1$ , and  $gM = 1 + O(z^{-1})$ ,  $\det(M)$  is holomorphic on  $\Delta$ , and on  $S^1$  equals a function which is holomorphic in  $\Delta^*$  and equal to 1 at  $\infty$ . Consequently  $\det(M)$  has a holomorphic extension to all of  $\hat{\mathbb{C}}$ , and hence must be identically 1. We can now take  $g_0g_+ = M^{-1}$ . This will have  $L^2$  entries, because  $M$  is unimodular.

For part (b), suppose that  $g$  has Birkhoff factorization  $g = g_-g_0g_+$ , and let  $g_0 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ . The matrix representing  $M_{g_0g_+}$  has the form

$$\begin{array}{cccccc|cccc} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \alpha & \beta & * & * & | & * & * & \cdot & \cdot \\ \cdot & \gamma & \delta & * & * & | & * & * & \cdot & \cdot \\ \cdot & 0 & 0 & \alpha & \beta & | & * & * & \cdot & \cdot \\ \cdot & 0 & 0 & \gamma & \delta & | & * & * & \cdot & \cdot \\ \hline \cdot & 0 & 0 & 0 & 0 & | & \alpha & \beta & \cdot & \cdot \\ \cdot & 0 & 0 & 0 & 0 & | & \gamma & \delta & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & | & \cdot & \cdot & \cdot & \cdot \end{array}$$

The matrix representing  $M_{g_-}$  is unipotent and lower triangular. Consequently  $A_1(g) = A_1(g_-)A_1(g_0g_+)$ ,  $A_1(g_-)$  is unipotent lower triangular, and  $A_1(g)$  is invertible iff  $A_1(g_0g_+)$  is invertible iff  $\alpha = (g_0)_{11} \neq 0$ . This implies Part (b).  $\square$

In Theorem 1.1 we are assuming that  $g$  is bounded. It is not generally true that the factors  $g_{\pm}$  are bounded. Recall (see [2]) that a Banach  $*$ -algebra  $\mathbb{A} \subset L^\infty(S^1)$  is said to be decomposing if

$$\mathbb{A} = \mathbb{A}_+ \oplus \mathbb{A}_-,$$

i.e.  $P_+ : \mathbb{A} \rightarrow \mathbb{A}_+$  is continuous. For example  $C^s(S^1)$  is decomposing, provided  $s > 0$  and nonintegral (see page 60 of [2]), and  $W^s$  is a decomposing algebra, provided  $s > 1/2$  (Note:  $W^{1/2}$  is not an algebra).

**Corollary 1.** *Suppose that  $g \in L^\infty(S^1, SL(2, \mathbb{C}))$  belongs to a decomposing algebra  $\mathbb{A}$  and has a Birkhoff factorization. Then the factors  $g_{\pm}$  belong to  $\mathbb{A}$ .*

This follows from the continuity of  $P_+$  on  $\mathbb{A}$  and the formula in (a) of Theorem 1.1.

**Theorem 1.2.** *If  $g \in L^\infty(S^1, SL(2, \mathbb{C}))$ , then  $B(g)$  and  $C(g)$  are compact operators if and only if  $g \in VMO$ , the space of functions with vanishing mean oscillation. If  $g \in QC := L^\infty \cap VMO$ , then  $A(g)$  and  $D(g)$  are Fredholm of index 0.*

The first statement is due to Hartmann, and the second to Douglas (see pages 27 and 108 of [3], respectively).

**Remarks.** (a) In the context of Theorem 1.1, if  $g$  has a Birkhoff factorization, then  $A(g)$  is 1 – 1: for if  $h \in \mathcal{H}_+$ , then there is a Hardy decomposition of (not necessarily  $L^2$ )  $\mathbb{C}^2$  valued functions

$$g_-^{-1}(M_g h)_+ = g_0g_+h - g_-^{-1}(M_g h)_-;$$

thus if  $A(g)h = 0$ , then  $h = 0$ . A Birkhoff factorization for bounded  $g$  does not imply  $A(g)$  is invertible (see Theorem 5.1, page 109 of [3]).

(b) For  $g \in QC(S^1, SL(2, \mathbb{C}))$ , the converse in (a) (and also (b)) of Theorem 1.1 holds, because the Fredholm index of  $A(g)$  vanishes. Moreover there is a notion of generalized triangular factorization for all  $g$  (see [2] and chapter 8 of [6]).

(c) Theorem 1.2 implies that the Toeplitz operator defines a holomorphic map

$$QC(S^1, SL(2, \mathbb{C})) \rightarrow \text{Fred}(\mathcal{H}_+) : g \rightarrow A(g).$$

There is a determinant line bundle  $\text{Det} \rightarrow \text{Fred}(\mathcal{H}_+)$  with canonical section,  $A \rightarrow \det(A)$ , which is nonvanishing precisely when  $A$  is invertible. In the notation of [4],  $\sigma_0 = \det(A(\tilde{g}))$  is the pullback of the canonical section, and  $\sigma_1 = \det(A(\sigma(\tilde{g})))$ , viewed as holomorphic functions of  $\tilde{g}$  in the universal  $\mathbb{C}^*$  extension of  $QC(S^1, SL(2, \mathbb{C}))$ . If  $g$  has a triangular factorization, then

$$(1.4) \quad m(g)a(g) = \begin{pmatrix} \sigma_1/\sigma_0 & 0 \\ 0 & \sigma_0/\sigma_1 \end{pmatrix},$$

as the matrix manipulations above suggest (see (1.5)-(1.6) of [4]).

## 2. PROOF OF THEOREM 0.1, AND GENERALIZATIONS TO OTHER FUNCTION SPACES

In the course of proving Theorem 0.1, we will also prove the following

**Theorem 2.1.** *Suppose that  $k_1 \in C^s(S^1, SU(2))$ , where  $s > 0$  and nonintegral. The following are equivalent:*

(a<sub>1</sub>)  $k_1$  is of the form

$$k_1(z) = \begin{pmatrix} a(z) & b(z) \\ -b^* & a^* \end{pmatrix}, \quad z \in S^1,$$

where  $a, b \in H^0(\Delta)$  have  $C^s$  boundary values,  $a(0) > 0$ , and  $a$  and  $b$  do not simultaneously vanish at a point in  $\Delta$ .

(c<sub>1</sub>)  $k_1$  has triangular factorization of the form

$$\begin{pmatrix} 1 & 0 \\ \sum_{j=0}^n y_j^* z^{-j} & 1 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix} \begin{pmatrix} \alpha_1(z) & \beta_1(z) \\ \gamma_1(z) & \delta_1(z) \end{pmatrix},$$

where the factors have  $C^s$  boundary values.

Similarly, the following are equivalent:

(a<sub>2</sub>)  $k_2$  is of the form

$$k_2(z) = \begin{pmatrix} d^* & -c^* \\ c(z) & d(z) \end{pmatrix}, \quad z \in S^1,$$

where  $c, d \in H^0(\Delta)$  have  $C^s$  boundary values,  $c(0) = 0$ ,  $d(0) > 0$ , and  $c$  and  $d$  do not simultaneously vanish at a point in  $\Delta$ .

(c<sub>2</sub>)  $k_2$  has triangular factorization of the form

$$\begin{pmatrix} 1 & \sum_{j=1}^n x_j^* z^{-j} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 & 0 \\ 0 & a_2^{-1} \end{pmatrix} \begin{pmatrix} \alpha_2(z) & \beta_2(z) \\ \gamma_2(z) & \delta_2(z) \end{pmatrix},$$

where the factors have  $C^s$  boundary values.

**Remarks.** (a) When  $k_2 \in L_{fin}SU(2)$ , the determinant condition  $c^*c + dd^* = 1$  can be interpreted as an equality of finite Laurent expansions in  $\mathbb{C}^*$ . Together with  $d(0) > 0$ , this implies that  $c$  and  $d$  do not simultaneously vanish. Thus the added hypotheses in (a<sub>i</sub>) of Theorem 2.1 are superfluous in the finite case.

(b) The kind of example we have to avoid in the  $C^\infty$  case is

$$k_2 = \begin{pmatrix} d^* & 0 \\ 0 & d \end{pmatrix}, \quad d = \frac{z-r}{rz-1}$$

where  $0 < r < 1$ .

(c) The factorizations in (b<sub>i</sub>) of Theorem 0.1 are akin to nonabelian Fourier expansions. Consequently it is highly unlikely that one can characterize the coefficients for  $C^s$  loops. For this purpose we consider a Sobolev completion at the end of this section.

*Proof.* As we remarked in the Introduction, the two sets of conditions are intertwined by the outer involution  $\sigma$ . Also it is evident that  $(c_2) \implies (a_2)$ : by multiplying the matrices in  $(c_2)$ , we see that  $c = a_2^{-1}\gamma_2$  and  $d = a_2^{-1}\delta_2$ , and these cannot simultaneously vanish at a point in  $\Delta$ . We will now prove, in reference to Theorem 0.1, that  $(b_2) \implies (a_2) \implies (c_2) \implies (b_2)$ . The second step will also complete the proof of Theorem 2.1.

It is straightforward to calculate that a loop as in  $(b_2)$  has the matrix form in  $(a_2)$ :

**Proposition 2.** *The product in  $(b_2)$  equals*

$$\left( \prod a(\zeta_i) \right) \begin{pmatrix} \delta_2^* & -\gamma_2^* \\ \gamma_2 & \delta_2 \end{pmatrix},$$

where

$$\gamma_2(z) = \sum_{n=1}^{\infty} \gamma_{2,n} z^n,$$

$$\gamma_{2,n} = \sum (-\bar{\zeta}_{i_1})\zeta_{j_1} \dots (-\bar{\zeta}_{i_r})\zeta_{j_r} (-\bar{\zeta}_{i_{r+1}}),$$

the sum over multiindices satisfying

$$0 < i_1 < j_1 < \dots < j_r < i_{r+1}, \quad \sum i_* - \sum j_* = n,$$

and

$$\delta_2(z) = 1 + \sum_{n=1}^{\infty} \delta_{2,n} z^n,$$

$$\delta_{2,n} = \sum \zeta_{i_1} (-\bar{\zeta}_{j_1}) \dots \zeta_{i_r} (-\bar{\zeta}_{j_r}),$$

the sum over multiindices satisfying

$$0 < i_1 < j_1 < \dots < j_r, \quad \sum (j_* - i_*) = n.$$

This is a straightforward induction, which we omit.

Now suppose that we are given a loop  $k_2$  satisfying the conditions in  $(a_2)$ , with one exception: for later convenience, we initially assume that  $k_2$  is merely measurable. Suppose that

$$A(k_2)f = P_+ \left( \begin{pmatrix} d^* & -c^* \\ c & d \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then  $cf_1 + df_2 = 0 \in H^0(\Delta)$ , and hence by the independence of  $c$  and  $d$  around  $S^1$ ,  $(f_1, f_2) = \lambda(d, -c)$ . Because  $c$  and  $d$  do not simultaneously vanish, this implies that  $\lambda$  is holomorphic in  $\Delta$ . We also have  $(d^*\lambda d - c^*\lambda(-c))_+ = \lambda_+ = 0$ . Thus  $\lambda = 0$ . Thus the Toeplitz operator is invertible [Note: conversely, if  $c$  and  $d$  have a common zero  $z_0 \in \Delta$ , then the Toeplitz operator is not invertible: take  $\lambda = 1/(z - z_0)$ ]. The same argument shows that  $A_1(k_2)$ , and also  $D(k_2)$ , are invertible.

We must now show that this loop has a triangular factorization as in  $(c_2)$ , i.e. we must solve for  $a_2$ ,  $x^*$ , and so on, in

$$(2.1) \quad k_2(z) = \begin{pmatrix} d^* & -c^* \\ c(z) & d(z) \end{pmatrix} = \begin{pmatrix} 1 & \sum_{j=1}^n \bar{x}_j z^{-j} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 & 0 \\ 0 & a_2^{-1} \end{pmatrix} \begin{pmatrix} \alpha_2(z) & \beta_2(z) \\ \gamma_2(z) & \delta_2(z) \end{pmatrix}.$$

The form of the second row implies that we must have  $a_2 = d(0)^{-1}$ , and

$$(2.2) \quad \gamma_2 = a_2 c, \quad \text{and} \quad \delta_2 = a_2 d.$$

because  $\delta_2(0) = 1$ . This does define  $a_2 > 0$ ,  $\gamma_2$  and  $\delta_2$  in a way which is consistent with  $(c_2)$ , because  $c(0) = 0$  and  $d(0) > 0$ .

Using (2.2), the first row in (2.1) is equivalent to

$$(2.3) \quad d^* = \alpha_2 + x^* c, \quad \text{and} \quad -c^* = \beta_2 + x^* d$$

In the finite case, by considering the second equation as an equality in  $\mathbb{C}^*$ , we can immediately obtain that  $x^* = -(c^*/d)_-$ . The  $C^s$  case is more involved.

Consider the Hardy space polarization

$$H := L^2(S^1, d\theta) = H^+ \oplus H^-,$$

and the operator

$$T : H^- \rightarrow H^- \oplus H^- : x^* \rightarrow ((cx^*)_-, (dx^*)_-).$$

The operator  $T$  is the restriction of  $D(k_2)^* = D(k_2^*)$  to the subspace  $\{(x^*, 0) \in \mathcal{H}^-\}$ , consequently it is injective with closed image.

The adjoint of  $T$  is given by

$$T^* : H^- \oplus H^- \rightarrow H^- : (f^*, g^*) \rightarrow c^* f^* + d^* g^*.$$

If  $(f^*, g^*) \in \ker(T^*)$ , then  $c^* f^* + d^* g^*$  vanishes in the closure of  $\Delta^*$ , and because  $|c|^2 + |d|^2 = 1$  around  $S^1$ ,  $(f^*, g^*) = \lambda^*(d^*, -c^*)$ , where  $\lambda^*$  is holomorphic in  $\Delta^*$  and vanishes at  $\infty$  because  $d^*(\infty) = d(0) > 0$ . We now claim that  $(d_-^*, -c_-^*) \in \ker(T^*)^\perp$ :

$$\int ((d_-^*)f + (-c_-^*)g)d\theta = \int \lambda(d^*d + c^*c)d\theta = \int \lambda d\theta = 0,$$

because  $\lambda(0) = 0$ . Because  $T$  has closed image, there exists  $x^* \in H^-$  such that

$$(2.4) \quad d_-^* = (x^*c)_-, \quad \text{and} \quad -c_-^* = (x^*d)_-.$$

We can now solve for  $\alpha_2$  and  $\beta_2$  in (2.3). This shows that  $k_2$  in  $(a_2)$  has a triangular factorization as in  $(c_2)$ . When  $k_2 \in C^s$ , by Corollary 1, the factors are  $C^s$ . This completes the proof of Theorem 2.1.

We have now shown that  $(b_2) \implies (a_2) \implies (c_2)$ . To prove that  $(c_2)$  implies  $(b_2)$ , one method is to explicitly solve for  $x^*$  in terms of the  $\zeta$  variables, then show



that this relation can be inverted. The formula for  $x$  in terms of  $\zeta$  is discussed in the Appendix. For our present purposes we only need to know that

$$x^* = \sum_{j=1}^{\infty} x_1^*(\zeta_j, \dots) z^{-j},$$

where

$$x_1^*(\zeta_1, \dots) = \zeta_1 \prod_{k=2}^{\infty} (1 + |\zeta_k|^2) + \zeta_2 \prod_{k=3}^{\infty} (1 + |\zeta_k|^2) s_2(\zeta_2, \zeta_3, \dots) + \zeta_3 \prod_{k=4}^{\infty} (1 + |\zeta_k|^2) s_3(\zeta_3, \zeta_4, \dots) + \dots$$

(in the current context, these are finite sums). This structure implies that we can solve for the  $\zeta_j$  in terms of the  $x_i$ , and in fact

$$\zeta_n(x_1, x_2, \dots) = \zeta_1(x_n, x_{n+1}, \dots).$$

(Note: the equivalence of  $(b_2)$  and  $(c_2)$  is implied by Theorem 5 of [4], which uses Lie theory; here we are emphasizing the elementary nature of the correspondence). This completes the proof of Theorem 0.1.  $\square$

It is obvious that for  $k_2$  in Theorem 0.1, there is a factorization  $a_2 = \prod a(\zeta_j)^{-1}$ . By considering the Kac-Moody central extension of  $LSU(2)$ , one can obtain a refinement of this factorization (recall (1.4), which suggests the existence of this refinement).

**Theorem 2.2.** *For  $k_i$  as in Theorem 0.1,  $\det(A^* A(k_1))$  equals*

$$\lim_{N \rightarrow \infty} \det(A_N(k_1)) = \det(1 - C^* C(k_1)) = \det(1 + \dot{B}^* \dot{B}(y))^{-1} = \prod_{n \geq 1} (1 + |\eta_n|^2)^{-n}$$

and  $\det(A^* A(k_2))$  equals

$$\lim_{N \rightarrow \infty} \det(A_N(k_2)) = \det(1 - C^* C(k_2)) = \det(1 + \dot{B}^* \dot{B}(x))^{-1} = \prod_{n \geq 1} (1 + |\zeta_n|^2)^{-n},$$

where  $A_N$  denotes the finite dimensional compression of  $A$  to the span of  $\{\epsilon_i z^k : 0 \leq k \leq N\}$ , and in the third expressions,  $x$  and  $y$  are viewed as multiplication operators on  $H = L^2(S^1)$ , with Hardy space polarization.

The first equalities are special cases of Theorem 6.1 of [7]; these are included for perspective: they demonstrate that finite dimensional approximations detect the magnitude of  $\det A$ , not its phase. The second equalities follow from the unitarity of the  $M_{k_i}$ ; they explain why the determinants are well-defined, since  $C(k_i)$  is Hilbert-Schmidt if and only if  $k_i \in W^{1/2}$  (this follows immediately from the matrix expression for  $M_{k_i}$  in Section 1). The last two equalities follow from Theorem 5 of [4].

**Lemma 1.** *Suppose that  $\zeta = (\zeta_n) \in l^2$ . As in Theorem 0.1, let*

$$k_2^{(N)} = \begin{pmatrix} d^{(N)*} & -c^{(N)*} \\ c^{(N)} & d^{(N)} \end{pmatrix} := \left( \prod_{n=1}^N a(\zeta_n) \right) \begin{pmatrix} 1 & \zeta_N z^{-N} \\ -\bar{\zeta}_N z^N & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & \zeta_1 z^{-1} \\ -\bar{\zeta}_1 z & 1 \end{pmatrix}.$$

Then  $c^{(N)}$  and  $d^{(N)}$  converge uniformly on compact subsets of  $\Delta$  to holomorphic functions  $c = c(\zeta)$  and  $d = d(\zeta)$ , respectively, as  $N \rightarrow \infty$ . The functions  $c$  and

$d$  have radial limits at a.e. point of  $S^1$ ,  $c$  and  $d$  are uniquely determined by these radial limits,

$$k_2(\zeta) := \begin{pmatrix} d(\zeta)^* & -c(\zeta)^* \\ c(\zeta) & d(\zeta) \end{pmatrix} \in Meas(S^1, GL(2, \mathbb{C})),$$

and  $\det(k_2) \leq 1$  on  $S^1$ .

A crucial lingering issue is the unitarity of  $k_2$ . In the course of proving Theorem 2.3, we will prove that  $k_2$  is unitary on  $S^1$  when  $\zeta \in w^{1/2}$ . Conjecturally this is true for  $\zeta \in l^2$  (see Section 4).

*Proof.* Because  $d^{(N)}d^{(N)*} + c^{(N)}c^{(N)*} = 1$ , both  $(c^{(N)})$  and  $(d^{(N)})$  are sequences of holomorphic functions on  $\Delta$  which are bounded by 1. By the Arzela-Ascoli Theorem, there exist subsequences which converge uniformly to holomorphic functions on  $\Delta$ , which will also be bounded by 1.

We claim these limits are unique. As in Proposition 2, write  $k^{(N)}$  as

$$\left( \prod_{n=1}^N a(\zeta_n) \right) \begin{pmatrix} \delta_2^{(N)*} & -\gamma_2^{(N)*} \\ \gamma_2^{(N)} & \delta_2^{(N)} \end{pmatrix}.$$

The  $\prod_{n=1}^\infty a(\zeta_n)$  converges, because  $\zeta \in l^2$ . Proposition 2 gives explicit expressions for the coefficients of  $\gamma_2^{(N)}$  and  $\delta_2^{(N)}$ . Very crude estimates show that these expressions have well-defined limits as  $N \rightarrow \infty$ . To see this, consider the formula for the  $n$ th coefficient of  $\delta_2$ , and let  $\mathcal{P}(n)$  denote the set of partitions of  $n$  (i.e. decreasing sequences  $n_1 \geq n_2 \geq \dots \geq n_l > 0$ , where  $\sum n_j = n$  is the magnitude and  $l = l(n_j)$  is the length of the partition). Then

$$(2.5) \quad |\delta_{2,n}| \leq \sum |\zeta_{i_1}| |\bar{\zeta}_{j_1}| \dots |\zeta_{i_r}| |\bar{\zeta}_{j_r}|,$$

where the sum is over multiindices satisfying

$$0 < i_1 < j_1 < \dots < j_r, \quad \sum (j_s - i_s) = n.$$

If  $n_k = j_k - i_k$ , then  $\sum n_k = n$ , but this sequence is not necessarily decreasing. However if we eliminate the constraints  $i_1 < \dots < i_r$ , then we can permute the indices ( $1 \leq k \leq r$ ) for the  $i_k$  and  $n_k$ . We can crudely estimate that (2.5) is

$$\begin{aligned} &\leq \sum_{(n_i) \in \mathcal{P}(n)} \sum_{i_1, \dots, i_l > 0} |\zeta_{i_1}| |\zeta_{i_1+n_1}| \dots |\zeta_{i_l}| |\zeta_{i_l+n_l}| = \sum_{(n_i) \in \mathcal{P}(n)} \prod_{s=1}^l \sum_{i_s > 0} |\zeta_{i_s}| |\zeta_{i_s+n_s}| \\ &\leq \sum_{\mathcal{P}(n)} |\zeta|_{l^2}^{2l((n_i))} \end{aligned}$$

This shows that the Taylor coefficients of any limiting function for the  $\delta^{(N)}$  will be given by the formulas in Proposition 2. The same considerations apply to the  $\gamma^{(N)}$ . Thus the sequences  $(\gamma^{(N)})$  and  $(\delta^{(N)})$  converge uniformly on compact sets of  $\Delta$  to unique limiting functions. This proves our claim about uniqueness of the limits  $c$  and  $d$ .

Because  $c$  and  $d$  are bounded by 1 on  $\Delta$ ,  $c$  and  $d$  have radial limits at a.e. point of  $S^1$ , and these boundary values uniquely determine  $c$  and  $d$ .

Finally we consider  $\det(k_2)$  on  $S^1$ . Since  $c$  and  $d$  are holomorphic in  $\Delta$ , and  $d(0) = \prod a(\zeta_j) \neq 0$ ,  $\det(k_2) = |d|^2 + |c|^2$  is nonzero a.e. on  $S^1$ . Thus  $k_2$  is invertible

a.e. on  $S^1$ . Clearly  $|d|^2 + |c|^2 \leq 2$  on the closure of  $\Delta$ , since  $|d|$  and  $|c|$  are bounded by 1. This also holds for  $d^{(N)}$  and  $c^{(N)}$ . If  $\rho \in L^1(S^1, d\theta)$  is positive, then

$$\int_{S^1} (|d|^2 + |c|^2) \rho d\theta = \lim_{r \uparrow 1} \int_{S^1} (|d|^2 + |c|^2) (re^{i\theta}) \rho(e^{i\theta}) d\theta,$$

(by dominated convergence)

$$\begin{aligned} &= \lim_{r \uparrow 1} \lim_{N \rightarrow \infty} \int_{S^1} (|d^{(N)}|^2 + |c^{(N)}|^2) (re^{i\theta}) \rho(e^{i\theta}) d\theta \leq \lim_{N \rightarrow \infty} \limsup_{r \uparrow 1} \int_{S^1} (|d^{(N)}|^2 + |c^{(N)}|^2) (re^{i\theta}) \rho(e^{i\theta}) d\theta \\ &= \lim_{N \rightarrow \infty} \int_{S^1} (|d^{(N)}|^2 + |c^{(N)}|^2) \rho(e^{i\theta}) d\theta = \int_{S^1} \rho(e^{i\theta}) d\theta \end{aligned}$$

Since  $\rho$  is a general positive integrable function, this implies that  $|d|^2 + |c|^2 \leq 1$  on  $S^1$ .

This completes the proof.  $\square$

*Remark.* To show that  $k_2$  has values in  $SU(2)$ , it would suffice to show

$$(2.6) \quad \frac{1}{2\pi} \int_{S^1} (|d|^2 + |c|^2) d\theta = 1.$$

This would follow immediately (by dominated convergence) if we knew that  $c^{(N)}$  ( $d^{(N)}$ ) converged to  $c$  ( $d$ , respectively) on  $S^1$ . But we have not shown this. Since  $d(0) = \prod a(\zeta_j)$ , it is clear that (2.6) is bounded below by  $\prod a(\zeta_j)^2$ .

**Theorem 2.3.** *Suppose that  $k_1 \in \text{Meas}(S^1, SU(2))$ . The following are equivalent:*

(a<sub>1</sub>)  $k_1$  is of the form

$$k_1(z) = \begin{pmatrix} a(z) & b(z) \\ -b^* & a^* \end{pmatrix}, \quad z \in S^1,$$

where  $a, b \in H^0(\Delta)$  have  $W^{1/2}$  boundary values,  $a(0) > 0$ , and  $a$  and  $b$  do not simultaneously vanish at a point in  $\Delta$ .

(b<sub>1</sub>)  $k_1$  has a factorization of the form

$$k_1(z) = \lim_{n \rightarrow \infty} a(\eta_n) \begin{pmatrix} 1 & -\bar{\eta}_n z^n \\ \eta_n z^{-n} & 1 \end{pmatrix} \cdot a(\eta_0) \begin{pmatrix} 1 & -\bar{\eta}_0 \\ \eta_0 & 1 \end{pmatrix},$$

where  $\eta \in w^{1/2}$ , and the limit is understood as in Lemma 1.

(c<sub>1</sub>)  $k_1$  has triangular factorization of the form

$$\begin{pmatrix} 1 & 0 \\ \sum_{j=0}^n y_j^* z^{-j} & 1 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix} \begin{pmatrix} \alpha_1(z) & \beta_1(z) \\ \gamma_1(z) & \delta_1(z) \end{pmatrix},$$

where  $y$  has  $W^{1/2}$  boundary values.

Moreover this defines a bijective correspondence between  $\eta \in w^{1/2}$  and  $(y_n) \in w^{1/2}$ .

Similarly, the following are equivalent:

(a<sub>2</sub>)  $k_2$  is of the form

$$k_2(z) = \begin{pmatrix} d^* & -c^* \\ c(z) & d(z) \end{pmatrix}, \quad z \in S^1,$$

where  $c, d \in H^0(\Delta)$  have  $W^{1/2}$  boundary values,  $c(0) = 0$ ,  $d(0) > 0$ , and  $c$  and  $d$  do not simultaneously vanish at a point in  $\Delta$ .

(b<sub>2</sub>)  $k_2$  has a factorization of the form

$$k_2(z) = \lim_{n \rightarrow \infty} a(\zeta_n) \begin{pmatrix} 1 & \zeta_n z^{-n} \\ -\bar{\zeta}_n z^n & 1 \end{pmatrix} \dots a(\zeta_1) \begin{pmatrix} 1 & \zeta_1 z^{-1} \\ -\bar{\zeta}_1 z & 1 \end{pmatrix},$$

where  $\zeta \in w^{1/2}$ , and the limit is understood as in Lemma 1.

(c<sub>2</sub>)  $k_2$  has triangular factorization of the form

$$\begin{pmatrix} 1 & \sum_{j=1}^n x_j^* z^{-j} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 & 0 \\ 0 & a_2^{-1} \end{pmatrix} \begin{pmatrix} \alpha_2(z) & \beta_2(z) \\ \gamma_2(z) & \delta_2(z) \end{pmatrix},$$

where  $x$  has  $W^{1/2}$  boundary values.

Moreover this defines a bijective correspondence between  $\zeta \in w^{1/2}$  and  $(x_n) \in w^{1/2}$ .

**Remarks.** (a) If  $\sum |\zeta_n| < \infty$ , then the products in (b<sub>i</sub>) converge absolutely and uniformly in  $z \in S^1$ , and the limits are  $C^0$ . However  $\sum n|\zeta_n|^2 < \infty$  does not imply absolute convergence of the sum of the  $\{\zeta_n\}$  and vice versa; similarly  $C^0$  does not imply  $W^{1/2}$  and vice versa. It is for this reason that the weak notion of convergence in Lemma 1 is used in (b<sub>i</sub>).

(b) In connection with (b<sub>i</sub>), note that  $z^n$  converges to zero uniformly on compact subsets of  $\Delta$ , but  $|z^n| = 1$ , for all  $n$ , on  $S^1$ . Thus it is not evident in (b<sub>i</sub>) that  $k_2$  is unitary; this is the problem which we could not resolve in Lemma 1.

*Proof.* The two sets of conditions are intertwined by  $\sigma$ . We will first show (a<sub>2</sub>) is equivalent to (c<sub>2</sub>); we will then show these conditions are equivalent to (b<sub>2</sub>).

Suppose that  $k_2$  satisfies the conditions in (a<sub>2</sub>), except that at the outset we only assume  $k_2$  is measurable. In the course of proving Theorem 2.1, we showed that  $k_2$  has a triangular factorization as in (c<sub>2</sub>), where

$$(2.7) \quad \begin{pmatrix} x^* \\ 0 \end{pmatrix} = D(k_2^*)^{-1} \begin{pmatrix} (d^*)_- \\ -c^* \end{pmatrix}$$

(and the other factors are given explicitly by (a) of Theorem 1.1). In particular  $x^* \in L^2$ .

For the Birkhoff factorization of  $k_2$ ,

$$(k_2)_- = \begin{pmatrix} 1 & x^* \\ 0 & 1 \end{pmatrix}.$$

Because  $M_{k_2}$  is unitary,

$$(2.8) \quad A(k_2)A(k_2)^* = (1 + Z(k_2)^* Z(k_2))^{-1},$$

where  $Z(k_2) := C(k_2)A(k_2)^{-1}$ . A matrix calculation (see (5.13) and (5.14) of [4], and note that in [4],  $g = k_2$ , and  $x$  is written in place of  $x^*$ ) shows that

$$(2.9) \quad Z(k_2) = Z((k_2)_-) = C((k_2)_-),$$

and relative to the basis (1.2),  $C((k_2)_-)$  is represented by the matrix

$$(2.10) \quad \begin{pmatrix} \cdot & 0 & x_n & \cdot & 0 & x_3 & 0 & x_2 & 0 & x_1 \\ \cdot & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & & x_4 & 0 & x_3 & 0 & x_2 & \\ & & & & 0 & 0 & 0 & 0 & 0 & \\ \cdot & & & & & & & 0 & x_3 & \\ \cdot & & \cdot & & \cdot & \cdot & & & & \\ \cdot & & & & \cdot & & & 0 & x_n & \\ 0 & 0 & 0 & 0 & \cdot & & 0 & 0 & 0 & \end{pmatrix}.$$

Now suppose that  $k_2 \in W^{1/2}$ . In this case  $A(k_2)A(k_2)^*$  is the identity plus trace class. By (2.8) and (2.9),  $C((k_2)_-)$  is Hilbert-Schmidt. By (2.10),  $x^* \in W^{1/2}$ .

Conversely, given  $x^* \in W^{1/2}$ , by Lemma 4 of [4], we can explicitly compute  $k_2$  and the corresponding triangular factorization:

$$(2.11) \quad \gamma_2 = -((1 + \dot{C}(zx^*)\dot{C}(zx^*)^*)^{-1}(x^*))^*, \quad \delta_2^* = 1 + \dot{C}(x^*)\gamma_2$$

$$(2.12) \quad \alpha_2 = a_2^{-2}(1 - \dot{A}(x^*)(\gamma_2)), \quad \beta = -a_2^{-2}\dot{A}(x^*)(\delta_2)$$

and

$$(2.13) \quad a_2^2 = \frac{\det(1 + \dot{C}(x^*)^*\dot{C}(x^*))}{\det(1 + \dot{C}(zx^*)^*\dot{C}(zx^*))}$$

In the derivation of the equations (2.11) and (2.12) in Lemma 4 of [4], the fact that  $k_2$  is unimodular is not used explicitly; the derivation only uses  $(k_2)_{(1,1)} = (k_2)_{(2,2)}^*$  and  $(k_2)_{(1,2)} = -(k_2)_{(2,1)}^*$ . However, because  $\alpha_2\delta_2 - \beta_2\gamma_2 \in H(\Delta)$ , and has real values  $|c|^2 + |d|^2$  on  $S^1$ ,  $\alpha_2\delta_2 - \beta_2\gamma_2$  extends holomorphically to  $\hat{\mathbb{C}}$ . Since it equals 1 at  $z = 0$ , it is identically 1. This shows that unimodularity follows automatically. This determines a unitary  $k_2$  with measurable coefficients. The calculations (2.8), (2.9), and (2.10) imply that  $k_2 \in W^{1/2}$ . Thus  $(a_2)$  is equivalent to  $(c_2)$ .

Lemma 1 implies that if  $(\zeta_n) \in l^2$ , then  $k_2$  defined as in  $(b_2)$  is in  $Meas(S^1, GL(2, \mathbb{C}))$ . Now suppose that  $\zeta \in w^{1/2}$ . By Theorem 2.2

$$(2.14) \quad \det|A(k_2^{(N)})|^2 = \det(1 + \dot{B}(x^{(N)})\dot{B}(x^{(N)})^*)^{-1} = \prod_{n=1}^N (1 + |\zeta_n|^2)^{-n},$$

and this converges to a positive number as  $N \rightarrow \infty$ .

First suppose that  $\zeta \geq 0$ . Proposition 4 of the Appendix implies that the coefficients of  $x(\zeta)^{(N)}$  are nonnegative and converge up to the coefficients of  $x(\zeta)$ . This implies that the matrix entries of  $B(x^{(N)})\dot{B}(x^{(N)})^*$  will be nonnegative and converge in a monotone way to those for  $\dot{B}(x)\dot{B}(x)^*$ . Thus the sequence  $tr(B(x^{(N)})\dot{B}(x^{(N)})^*)$ , which is bounded because (2.14) converges, will converge to  $tr(\dot{B}(x)\dot{B}(x)^*)$ . This implies that  $(x_n) \in w^{1/2}$ . For a general  $\zeta \in w^{1/2}$ , since the coefficients for  $x(|\zeta|)$  dominate those for  $x(\zeta)$  we can conclude in the same way that  $(x_n) \in w^{1/2}$ . We can now obtain a triangular factorization for  $k_2$  using (2.11)-(2.13). As we argued in the paragraph following (2.13), this automatically implies that  $k_2$  is unitary. The calculations (2.8), (2.9), and (2.10) imply that  $k_2 \in W^{1/2}$  and  $A(k_2)$  is invertible. Since  $A(k_2)$  is  $1 - 1$ , this implies that  $c$  and  $d$  do not simultaneously vanish in  $\Delta$  (see the Note in the second paragraph following Proposition 2). Thus  $(b_2)$  implies  $(a_2)$ .

Suppose that we are given  $k_2$  and  $x$  as in  $(a_2)$  and  $(c_2)$ . Let  $x^{(N)} = \sum_{n=1}^N x_n z^n$ , and let  $\zeta^{(N)}$  and  $k_2^{(N)}$  denote the corresponding objects. Theorem 2.2 implies that

$$(2.15) \quad \det(1 + \dot{B}(x^{(N)})\dot{B}(x^{(N)})^*) = \prod_{n=1}^N (1 + |\zeta_n^{(N)}|^2)^n.$$

Because  $x \in W^{1/2}$ , the sequence of numbers (2.15) has a limit. Therefore the sequence  $\{\zeta^{(N)}\}$  is bounded in  $w^{1/2}$ . Because the inclusion  $w^{1/2} \rightarrow l^2$  is a compact operator, there are subsequences which converge in  $l^2$ . By Lemma 1 these limiting sequences correspond to  $k_2$ . Thus there is a unique limiting sequence,  $\{\zeta_n\} \in l^2$ . Since (2.15) has a limit,  $\zeta \in w^{1/2}$ . Thus  $(a_2)$  and  $(c_2)$  imply  $(b_2)$ .

This completes the proof.  $\square$

### 3. PROOF OF THEOREM 0.2, AND GENERALIZATIONS

Part (a) of Theorem 0.2 is obvious. We will deduce the remaining parts of Theorem 0.2 from the following

**Theorem 3.1.** *Assume  $s > 0$  and nonintegral, or  $s = \infty$ . For  $g \in C^s(S^1, SU(2))$ , the following are equivalent:*

- (i)  *$g$  has a triangular factorization  $g = lmau$ , where  $l$  and  $u$  have  $C^s$  boundary values.*
- (ii)  *$g$  has a factorization  $g = k_1^* \lambda k_2$ , where the  $k_i \in C^s(S^1, SU(2))$  satisfy the equivalent conditions  $(a_i)$  and  $(c_i)$  of Theorem 2.1, and  $\lambda \in C^s(S^1, T)_0$ .*

*Proof.* We will use the notation in (1.1) for  $g$ , and the notation in Theorem 2.1 for the entries of the  $k_i$  and their triangular factorizations. Without much comment, we will use the fact that  $C^s$  is a decomposing algebra, so that factors in various decompositions will remain in  $C^s$ .

We proved that (ii) implies (i) in [4] (see the proof of Theorem 7); we briefly recall the calculation. Suppose that  $g \in C^s(S^1, SU(2))$  can be factored as  $g = k_1^* \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} k_2$ , as in (ii). We can write  $\lambda = \exp(-\chi^* + \chi_0 + \chi)$ , where  $\chi_0 \in i\mathbb{R}$  and  $\chi \in H^0(\Delta)$ ,  $\chi(0) = 0$ , with  $C^s$  boundary values. Then  $g$  has triangular factorization of the form

$$(3.1) \quad g = l(g) \begin{pmatrix} e^{\chi_0} a_1 a_2 & 0 \\ 0 & (e^{\chi_0} a_1 a_2)^{-1} \end{pmatrix} u(g),$$

where  $m_0 = e^{\chi_0} \in S^1$ ,  $a_0 = a_1 a_2 > 0$ ,

$$(3.2) \quad l(g) := \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} = \begin{pmatrix} \alpha_1^* & \gamma_1^* \\ \beta_1^* & \delta_1^* \end{pmatrix} \begin{pmatrix} e^{-\chi^*} & 0 \\ 0 & e^{\chi^*} \end{pmatrix} \begin{pmatrix} 1 & a_1^2 e^{2\chi_0} P_-(ye^{2\chi^*} + x^* e^{2\chi}) \\ 0 & 1 \end{pmatrix}$$

and

$$(3.3) \quad u(g) := \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} 1 & a_2^{-2} e^{-2\chi_0} P_+(ye^{2\chi} + x^* e^{2\chi^*}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^\chi & 0 \\ 0 & e^{-\chi} \end{pmatrix} \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix}$$

Thus (i) is implied by (ii).

Now suppose that  $g$  has triangular factorization  $g = lmau$  as in (i). We must solve for  $k_1$ ,  $\chi$ , and  $k_2$ . The equation (3.2) implies

$$(3.4) \quad l_{11} = \alpha_1^* \exp(-\chi^*), \quad l_{21} = \beta_1^* \exp(-\chi^*)$$

and (3.3) implies

$$(3.5) \quad u_{21} = \gamma_2 \exp(-\chi), \quad u_{22} = \delta_2 \exp(-\chi)$$

The special forms of  $k_1$  and  $k_2$  imply that on  $S^1$ ,

$$(3.6) \quad |\alpha_1|^2 + |\beta_1|^2 = a_1^{-2}.$$

$$(3.7) \quad |\delta_2|^2 + |\gamma_2|^2 = a_2^2.$$

Therefore on  $S^1$

$$(3.8) \quad |l_{11}|^2 + |l_{21}|^2 = a_1^{-2} \exp(-2\operatorname{Re}(\chi))$$

$$(3.9) \quad |u_{21}|^2 + |u_{22}|^2 = a_2^2 \exp(-2\operatorname{Re}(\chi))$$

This implies that on  $S^1$  we must have

$$(3.10) \quad \operatorname{Re}(\chi) = \log(a_1^{-1}) + \log((|l_{11}|^2 + |l_{21}|^2)^{-1/2}) = \log(a_2) + \log((|u_{21}|^2 + |u_{22}|^2)^{-1/2}).$$

Assuming that the obvious consistency condition is satisfied, this pair of equations determines  $\chi$  and the  $a_i$ : because  $\chi$  must be holomorphic in the disk and vanish at  $z = 0$ , the average of  $\operatorname{Re}(\chi)$  around  $S^1$  must vanish, hence

$$(3.11) \quad a_1 = \exp\left(-\frac{1}{4\pi} \int_{S^1} \log(|l_{11}|^2 + |l_{21}|^2) d\theta\right),$$

$$(3.12) \quad a_2 = \exp\left(\frac{1}{4\pi} \int_{S^1} \log(|u_{21}|^2 + |u_{22}|^2) d\theta\right),$$

and

$$(3.13) \quad \operatorname{Im}(\chi) = i\operatorname{Re}(\chi)_- - i\operatorname{Re}(\chi)_+.$$

To see that  $\chi$  and the  $a_i$  are well-defined, we must check that

$$(3.14) \quad |l_{11}|^2 + |l_{21}|^2 = (a_1 a_2)^{-2} (|u_{21}|^2 + |u_{22}|^2),$$

as functions on  $S^1$ . Because  $g^*g = 1$ ,  $l^*l = (a(g)u)^{-*}(a(g)u)^{-1}$ , on  $S^1$ . This implies three independent equations

$$(3.15) \quad |l_{11}|^2 + |l_{21}|^2 = a_0^{-2} (|u_{22}|^2 + |u_{21}|^2)$$

$$(3.16) \quad l_{11}^* l_{12} + l_{21}^* l_{22} = -(u_{22}^* u_{12} + u_{21}^* u_{11})$$

$$(3.17) \quad |l_{12}|^2 + |l_{22}|^2 = a_0^2 (|u_{12}|^2 + |u_{11}|^2)$$

for the  $(1, 1)$ ,  $(1, 2)$  (or  $(2, 1)$ ), and  $(2, 2)$  entries, respectively. The  $(1, 1)$  entry implies the consistency condition (3.14).

Together with (3.4) and (3.5), this completely determines the  $k_i$ :

$$(3.18) \quad a(z) = a_1 \exp(\chi) l_{11}^*, \quad b(z) = a_1 \exp(\chi) l_{21}^*,$$

$$(3.19) \quad c(z) = a_2^{-1} \exp(\chi) u_{21}, \quad d(z) = a_2^{-1} \exp(\chi) u_{22}$$

Because  $l^*$  is invertible at all points of  $\Delta$ , the entries  $a$  and  $b$  of  $k_1$  do not simultaneously vanish. Similarly, because  $u$  is invertible, the entries  $c$  and  $d$  do not simultaneously vanish. The fact that these are  $C^s$  in the appropriate sense follows from the continuity of the projections  $P_\pm$  on  $C^s$ . Thus by Theorem 2.1 (and the ensuing Remark (b)) the  $k_i$  have appropriate triangular factorizations.

We have now solved for  $k_i$  and  $\chi$ . We have also observed that the diagonal term of  $g$  determines  $\exp(\chi_0)$ , so  $\lambda$  is determined as well.

We now must show that  $g = k_1^{-1}\lambda k_2$ . From the definitions of  $k_i$  and  $\lambda$ , both sides of this equation have the same  $m$ ,  $a$ ,  $l_{11}$ ,  $l_{21}$ ,  $u_{21}$ , and  $u_{22}$  coordinates. The proof is completed by the next Proposition, which is of intrinsic interest.  $\square$

**Proposition 3.** *Suppose that  $g$  has a triangular factorization as in (1.1). Then*

$$\begin{aligned} l_{12} &= -l_{11}P_-\left(\frac{l_{21}^* + u_{21}^*}{|l_{11}|^2 + |l_{21}|^2}\right) \\ l_{22} &= 1 - l_{21}P_-\left(\frac{l_{21}^* + u_{21}^*}{|l_{11}|^2 + |l_{21}|^2}\right) \\ u_{12} &= -a_0^{-2}u_{22}P_+\left(\frac{l_{21}^* + u_{21}^*}{|l_{11}|^2 + |l_{21}|^2}\right) \\ u_{11} &= 1 - a_0^{-2}u_{21}P_+\left(\frac{l_{21}^* + u_{21}^*}{|l_{11}|^2 + |l_{21}|^2}\right) \end{aligned}$$

*In particular  $g$  is determined by  $m$ ,  $a$ ,  $l_{11}$ ,  $l_{21}$ ,  $u_{21}$ , and  $u_{22}$ .*

*Proof.* We initially suppose that  $l_{11}$  and  $u_{22}$  are nonvanishing. We can use the unimodularity of  $l$  and  $u$  to solve for  $l_{22}$  and  $u_{11}$  in terms of  $l_{12}$  and  $u_{12}$ .

The equation (3.16) can be rewritten as

$$\begin{aligned} l_{11}^*l_{12} + l_{21}^*l_{22} + u_{22}^*u_{12} + u_{21}^*u_{11} = \\ l_{11}^*l_{12} + l_{21}^*\left(1 + \frac{l_{12}l_{21}}{l_{11}}\right) + u_{22}^*u_{12} + u_{21}^*\left(1 + \frac{u_{12}u_{21}}{u_{22}}\right) = 0 \end{aligned}$$

Using (3.15) this can be rewritten as

$$\frac{l_{12}}{l_{11}} + a_0^2 \frac{u_{12}}{u_{22}} = -\frac{l_{21}^* + u_{21}^*}{|l_{11}|^2 + |l_{21}|^2}$$

by applying  $P_\pm$  to this equation, and solving, we obtain the equations in the proposition.  $\square$

Suppose that  $g \in C^s(S^1, SU(2))$ ,  $s > 1/2$ , and  $g$  has a triangular factorization. By Theorem 7 of [4],

$$\begin{aligned} (3.20) \quad \det(A^*A(g)) &= \det(A^*A(k_1^{-1}))\det(A^*A(\lambda))\det(A^*A(k_2)) \\ &= \prod_{i=1}^{\infty} (1 + |\eta_i|^2)^{-i} \exp(-2 \sum_{j=1}^{\infty} j |\chi_j|^2) \prod_{k=1}^{\infty} (1 + |\zeta_k|^2)^{-k}. \end{aligned}$$

These expressions make sense because  $C^s \subset W^{1/2}$  for  $s > 1/2$ . In the remainder of this section, our goal is to use these equalities to obtain a  $W^{1/2}$  analogue of Theorem 3.1, which also incorporates the condition  $(b_i)$ . This involves some subtleties, because  $W^{1/2}$  functions are not necessarily continuous.

Because  $SU(2)$  is compact,  $W^{1/2}(S^1, SU(2))$  is a separable topological group. In contrast to the function spaces  $C^s$ ,  $s > 0$ ,  $W^s$ ,  $s > 1/2$ , and  $L^\infty \cap W^{1/2}$ , for the function space  $W^{1/2}$ , the loop group  $W^{1/2}(S^1, SU(2))$  is not a Lie group, because



$W^{1/2}(S^1, su(2))$  is not a Lie algebra (whereas, e.g.  $L^\infty \cap W^{1/2}(S^1, su(2))$  has a Lie algebra structure). Moreover the inclusion  $C^\infty(S^1, SU(2)) \subset W^{1/2}(S^1, SU(2))$  is dense and presumably a homotopy equivalence (whereas this is false for the  $L^\infty \cap W^{1/2}$  topology). With respect to the  $W^{1/2}$  topology, the operator-valued function

$$g \rightarrow \begin{pmatrix} A(g) & B(g) \\ C(g) & D(g) \end{pmatrix}$$

is continuous, provided the diagonal is equipped with the strong operator topology, and the off-diagonal with the Hilbert-Schmidt topology.

In reference to the following Lemma, we recall that the notion of degree (or winding number) can be extended from  $C^0$  to  $VMO(S^1, S^1)$ , hence degree is well-defined for  $W^{1/2}(S^1, S^1)$  (see Section 3 of [1] for an amazing variety of formulas, and further references, or pages 98-100 of [3]). Also given  $\lambda \in W^{1/2}(S^1, S^1)$ , we view  $\lambda$  as a multiplication operator on  $H = L^2(S^1)$ , with the Hardy polarization. We write  $\dot{A}(\lambda)$  for the Toeplitz operator, and so on (with the dot), to avoid confusion with the matrix case.

**Lemma 2.** *There is an exact sequence of topological groups*

$$0 \rightarrow 2\pi i\mathbb{Z} \rightarrow W^{1/2}(S^1, i\mathbb{R}) \xrightarrow{\exp} W^{1/2}(S^1, S^1) \xrightarrow{\text{degree}} \mathbb{Z} \rightarrow 0.$$

Moreover  $\text{degree}(\lambda) = -\text{index}(\dot{A}(\lambda))$ .

There is a more general version of this involving  $VMO$ , which is implicit on pages 100-101 of [3].

*Proof.* Suppose that  $f \in W^{1/2}(S^1, i\mathbb{R})$ . It is convenient to use the equivalent Besov form of the  $W^{1/2}$  norm,

$$|f|_{W^{1/2}}^2 = \int \int \frac{|f(\theta_1) - f(\theta_2)|^2}{|e^{i\theta_1} - e^{i\theta_2}|^2} d\theta_1 d\theta_2.$$

Because  $|e^{i\theta} - 1| \leq |\theta|$ ,

$$\int \int \frac{|e^{f(\theta_1)} - e^{f(\theta_2)}|^2}{|e^{i\theta_1} - e^{i\theta_2}|^2} d\theta_1 d\theta_2 \leq |f|_{W^{1/2}}^2.$$

Thus  $\exp(f)$  is also  $W^{1/2}$ . This inequality also shows that  $\exp$  is continuous at 0. Since  $\exp$  is a homomorphism, this implies  $\exp$  is globally continuous.

Continuity implies that the image of  $\exp$  is contained in the identity component. Conversely suppose that  $\lambda \in W^{1/2}(S^1, S^1)_0$ . Then  $\dot{A}(\lambda)$  is invertible. This implies the existence of a Birkhoff factorization  $\lambda = \lambda_- \lambda_0 \lambda_+$ , where for example  $\lambda_+ \in H^0(\Delta, 0; \mathbb{C}^*, 1)$  and has  $L^2$  boundary values. By taking logarithms on the disks, we can write  $\lambda = \exp(-\chi^* + \chi_0 + \chi)$ . By a formula of Szego and Widom (Theorem 7.1 of [7]),

$$(3.21) \quad \det(\dot{A}^* \dot{A}(\lambda)) = \det(1 - \dot{C}^* \dot{C}(\lambda)) = \exp(-2 \sum_{j=1}^{\infty} j |\chi_j|^2)$$

The determinant depends continuously on  $\lambda$  in the  $W^{1/2}$  topology. Therefore  $\chi \in W^{1/2}$ . This shows the sequence is exact at  $W^{1/2}(S^1, S^1)$ .

A  $W^{1/2}$  function cannot have jump discontinuities. This implies that the kernel of  $\exp$  is  $2\pi i\mathbb{Z}$ . Thus the sequence in the statement of the Lemma is continuous and exact.  $\square$

**Theorem 3.2.** *For  $g \in W^{1/2}(S^1, SU(2))$ , the following are equivalent:*

- (i)  *$g$  has a triangular factorization  $g = lmau$ .*
  - (ii)  *$g$  has a factorization  $g = k_1^* \lambda k_2$ , where the  $k_i \in W^{1/2}(S^1, SU(2))$  satisfy the equivalent conditions of Theorem 2.3, and  $\lambda \in W^{1/2}(S^1, T)_0$ .*
- In both cases the factorization is unique.*

*Proof.* Given Lemma 2, the proof that (ii) implies (i) is the same as in the proof of Theorem 3.1.

Now assume (i). We can again solve for  $k_i$  and  $\chi$ , as in the proof of Theorem 3.1. The determinant formulas (3.20) can be applied to  $g^{(N)} = k_1^{(N)} \exp(\chi^{(N)}) k_2^{(N)}$ , where the subscript indicates that  $\zeta_n, \chi_n, \eta_n$  are set equal to 0, for  $n > N$ . In (3.20), applied to  $g^{(N)}$ , all of the individual factors in (3.20) are bounded above by 1, and are tending monotonically down. Since  $g \in W^{1/2}$ ,  $\det(A(g)A(g)^*)$  is positive, and  $\det(A(g^{(N)})A(g^{(N)})^*)$  will remain bounded away from zero. This implies that all of the factors in (3.20), applied to  $g^{(N)}$ , will be bounded away from 0. Thus  $\zeta, \chi$  and  $\eta$  are in  $w^{1/2}$ . By Theorem 2.3,  $k_i \in W^{1/2}$ . This implies (ii).  $\square$

**Corollary 2.** *The dense open set of  $g \in W^{1/2}(S^1, SU(2))$  having triangular factorization is parameterized by  $y, \chi_0 \in i\mathbb{R} \bmod 2\pi i\mathbb{Z}$ ,  $\chi$ , and  $x$ , where  $y, \chi$  and  $x$  are holomorphic functions in  $\Delta$  with  $W^{1/2}$  boundary values, and  $x(0) = \chi(0) = 0$ .*

*Remark.* This implies that an open neighborhood of  $1 \in W^{1/2}(S^1, SU(2))$  is parameterized by a Hilbert space. This should be compared to the finite dimensional situation, where a topological group locally homeomorphic to  $\mathbb{R}^n$  is automatically a  $C^\omega$  Lie group.

#### 4. A CONJECTURAL $L^2$ GENERALIZATION

Suppose that  $\zeta \in l^2$ . By Lemma 1 there is a unique limit  $k_2 \in Meas(S^1, GL(2, \mathbb{C}))$  for the product in (4.1) below. When  $A(k_2)$  is invertible, e.g. if  $\zeta \in w^{1/2}$  (by Theorem 2.3), there are three different expressions for  $k_2$ ,

$$(4.1) \quad \prod_{\leftarrow} a(\zeta_n) \begin{pmatrix} 1 & \zeta_n z^{-n} \\ -\bar{\zeta}_n z^n & 1 \end{pmatrix} = \left( \prod a(\zeta_n) \right) \begin{pmatrix} \delta_2^* & -\gamma_2^* \\ \gamma_2 & \delta_2 \end{pmatrix} = \begin{pmatrix} 1 & x^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 & 0 \\ 0 & a_2^{-1} \end{pmatrix} \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix},$$

where  $a_2 = \prod a(\zeta_j)^{-1}$ , and  $\gamma_2$  and  $\delta_2$  are determined by the formulas in Proposition 2. The existence of the triangular factorization implies that  $k_2$  has values in  $SU(2)$  on  $S^1$ .

Since the expression for  $a_2$  is convergent for all  $\zeta \in l^2$ , it is plausible that the triangular factorization in (4.1) is valid for all  $\zeta \in l^2$ . A further leap of faith suggests the following

**Conjecture.** *Suppose that  $k_2 \in Meas(S^1, SU(2))$ . The following are equivalent:*  
*(a<sub>2</sub>)  $k_2$  is of the form*

$$k_2(z) = \begin{pmatrix} d^* & -c^* \\ c(z) & d(z) \end{pmatrix}, \quad z \in S^1,$$

*where  $c, d \in H^0(\Delta)$ ,  $c(0) = 0$ ,  $d(0) > 0$ , and  $c$  and  $d$  do not simultaneously vanish at a point in  $\Delta$ .*

(b<sub>2</sub>)  $k_2$  has a factorization of the form

$$k_2(z) = \lim_{n \rightarrow \infty} a(\zeta_n) \begin{pmatrix} 1 & \zeta_n z^{-n} \\ -\bar{\zeta}_n z^n & 1 \end{pmatrix} \dots a(\zeta_1) \begin{pmatrix} 1 & \zeta_1 z^{-1} \\ -\bar{\zeta}_1 z & 1 \end{pmatrix},$$

where  $\zeta \in l^2$ , and the limit is understood as in Lemma 1.

(c<sub>2</sub>)  $k_2$  has triangular factorization of the form

$$\begin{pmatrix} 1 & \sum_{j=1}^n x_j^* z^{-j} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 & 0 \\ 0 & a_2^{-1} \end{pmatrix} \begin{pmatrix} \alpha_2(z) & \beta_2(z) \\ \gamma_2(z) & \delta_2(z) \end{pmatrix}.$$

Moreover this defines a bijective correspondence between  $\zeta \in l^2$  and  $(x_n) \in l^2$ .

In reference to this Conjecture, recall that the condition (a<sub>2</sub>) implies that  $A(k_2)$  is  $1-1$ . This entails invertibility when  $k_2 \in QC$  (see Theorem 1.2), but not in general. When  $k_2$  is expressed as in (c<sub>2</sub>), the third paragraph of the proof of Theorem 2.3, together with results of Nehari and Fefferman (pages 3-5 of [3]), implies that  $A(k_2)$  is invertible precisely when  $x$  has  $BMO$  boundary values. Thus the implications (b<sub>2</sub>)  $\implies$  (a<sub>2</sub>)  $\implies$  (c<sub>2</sub>) hinge on the question of whether  $\zeta \in l^2 \implies (x_n) \in l^2$ , and this is different from the question of when  $A(k_2)$  is invertible.

The implication (c<sub>2</sub>)  $\implies$  (a<sub>2</sub>) hinges on the formulas (2.11)-(2.13) for  $k_2$  in terms of  $x$ . The first two formulas make sense for  $x \in BMO$ , as in the preceding paragraph, but it is not clear that this is the natural domain for  $x$ . Regarding the formula for  $a_2$ , which a priori depends on  $(x_n) \in w^{1/2}$ , the second order term in the expansion at  $x = 0$  is

$$\text{tr}(C(x^*)C(x^*)^*) - \text{tr}(C(zx^*)C(zx^*)^*) = \sum |x_n|^2,$$

the  $l^2$  norm. This is at least consistent with the Conjecture.

## 5. APPENDIX. THE RELATION BETWEEN $x^*$ AND $\zeta$

In this Appendix, we consider the relation between  $x^*$  and  $(\zeta_j)$ , in Theorem 0.1, at the level of combinatorial formulas.

### 5.1. $x^*$ as a function of $\zeta$ .

**Proposition 4.**  $x^*$  has the form

$$x^* = \sum_{j=1}^{\infty} x_1^*(\zeta_j, \dots) z^{-j},$$

where

$$x_1^*(\zeta_1, \dots) = \sum_{n=1}^{\infty} \zeta_n \left( \prod_{k=n+1}^{\infty} (1 + |\zeta_k|^2) \right) s_n(\zeta_n, \zeta_{n+1}, \bar{\zeta}_{n+1}, \dots),$$

$s_1 = 1$  and for  $n > 1$ ,

$$s_n = \sum_{r=1}^{n-1} s_{n,r}, \quad s_{n,r} = \sum c_{i,j} \zeta_{i_1} \bar{\zeta}_{j_1} \zeta_{i_2} \bar{\zeta}_{j_2} \dots \zeta_{i_r} \bar{\zeta}_{j_r}$$

where the sum is over multiindices satisfying the constraints

$$(5.1) \quad \begin{array}{ccc} j_1 & \leq & \dots \leq j_r \\ \vee & & \vee \\ n & \leq & i_1 \leq \dots \leq i_r \end{array}, \quad \sum_{l=1}^r (j_l - i_l) = n - 1,$$

and  $c_{i,j}$  is a positive integer.

*Remark.* The main features of the formula for  $x_1^*$  are (i) the appearance of the infinite products, which isolates the part of the expression which has to be "renormalized" in probabilistic applications, and (ii) the positivity of the coefficients. For example (ii) implies that if  $\zeta \geq 0$ , then the coefficients for  $x(\zeta_1, \dots, \zeta_N, 0, \dots)$  converge monotonically up to the coefficients for  $x(\zeta)$  as  $N \rightarrow \infty$ .

*Proof.* The fact that  $x^*$  is completely determined by its residue  $x_1^*$  is (b) of Theorem 5 of [4]. We will show that  $x_1^*$  has the form claimed in the Lemma (I stated this without proof in [4]).

Clearly  $x_1^*(\zeta_1) = \zeta_1$ . The proof hinges on the following recursion (see Lemma 2 and (5.12) of [4])

$$\begin{aligned} x_1^*(\zeta_1, \dots, \zeta_{N+1}) = & (1 + |\zeta_{N+1}|^2) \{x_1(\zeta_1, \dots, \zeta_N) + \sum_{i+j=N+2} x_1(\zeta_i, \dots, \zeta_N) x_1(\zeta_j, \dots, \zeta_N)\} \bar{\zeta}_{N+1} \\ & + \sum_{i+j+k=2N+3} x_1(\zeta_i, \dots, \zeta_N) x_1(\zeta_j, \dots, \zeta_N) x_1(\zeta_k, \dots, \zeta_N) \bar{\zeta}_{N+1}^2 \\ & + \sum_{i+j+k+l=3N+4} x_1(\zeta_i, \dots, \zeta_N) x_1(\zeta_j, \dots, \zeta_N) x_1(\zeta_k, \dots, \zeta_N) x_1(\zeta_l, \dots, \zeta_N) \bar{\zeta}_{N+1}^3 + \dots \} \end{aligned}$$

From this recursion one can immediately see that coefficients will be nonnegative.

We assume that

$$x_1^*(\zeta_1, \dots, \zeta_N) = \sum_{n=1}^N \zeta_n \prod_{k=n+1}^N (1 + |\zeta_k|^2) s_n(\zeta_n, \dots, \zeta_N),$$

where  $s_1 = 1$  and for  $n > 1$

$$s_n(\zeta_n, \dots, \zeta_N) = \sum c_{i,j} \zeta_{i_1} \bar{\zeta}_{j_1} \zeta_{i_2} \bar{\zeta}_{j_2} \dots \zeta_{i_r} \bar{\zeta}_{j_r},$$

the sum is over multiindices as in (5.1), with  $j_r \leq N$ , and  $c_{i,j}$  is a positive integer (for  $N > 1$ ,  $s_N(\zeta_N) = 0$ ).

This implies

$$\begin{aligned} x_1^*(\zeta_I, \dots, \zeta_N) &= \sum_{n=1}^{N-(I-1)} \zeta_{n+(I-1)} \prod_{k=n+1}^{N-(I-1)} (1 + |\zeta_{k+(I-1)}|^2) s_n(\zeta_{n+(I-1)}, \dots) \\ &= \sum_{m=I}^N \zeta_m \prod_{k=m+1}^N (1 + |\zeta_k|^2) s_{m-(I-1)}(\zeta_m, \dots, \zeta_N) \end{aligned}$$

where

$$s_{m-(I-1)}(\zeta_m, \dots, \zeta_N) = \sum c_{i-(I-1)\vec{1}, j-(I-1)\vec{1}} \zeta_{i_1} \bar{\zeta}_{j_1} \dots \zeta_{i_L} \bar{\zeta}_{j_L},$$

the sum is over multiindices satisfying

$$\begin{array}{ccccccc} j_1 & \leq & \dots & \leq & j_L & \leq & N \\ \vee & & & & \vee & & \\ m & \leq & i_1 & \leq & \dots & \leq & i_L \end{array}, \quad \sum_{l=1}^L (j_l - i_l) = m - I,$$

and in the notation for the coefficient,  $i - (I-1)\vec{1}$  means that we subtract  $I-1$  from each of the components of  $i$ .

We now plug this into the recursion relation, and rewrite the expression so that it has the same form as the sum involving  $N$  variables:

$$\begin{aligned}
 (5.2) \quad x_1(\zeta_1, \dots, \zeta_{N+1}) &= (1 + |\zeta_{N+1}|^2) \sum_{s \geq 0} \left\{ \sum_{\sum_{l=1}^{s+1} I_l = s(N+1)+1} \prod_{I_l} x_1(\zeta_{I_l}, \dots, \zeta_N) \right\} \bar{\zeta}_{N+1}^s \\
 &= (1 + |\zeta_{N+1}|^2) \sum_{s \geq 0} \sum_{\sum_{l=1}^{s+1} I_l = s(N+1)+1} \prod_{I_l} \left( \sum_{m_l = I_l}^N \zeta_{m_l} \prod_{k=m_l+1}^N (1 + |\zeta_k|^2) s_{m_l - (I_l - 1)}(\zeta_{m_l}, \dots, \zeta_N) \right) \bar{\zeta}_{N+1}^s \\
 &= (1 + |\zeta_{N+1}|^2) \sum_{s \geq 0} \sum_{\sum_{l=1}^{s+1} I_l = s(N+1)+1} \sum_{m_1 = I_1}^N \dots \sum_{m_{s+1} = I_{s+1}}^N \\
 &\quad \prod_{I_l} [\zeta_{m_l} \prod_{k=m_l+1}^N (1 + |\zeta_k|^2) \sum c_{\vec{i}_l - (I_l - 1) \vec{1}, \vec{j}_l - (I_l - 1) \vec{1}} \bar{\zeta}_{i_{l,1}} \bar{\zeta}_{j_{l,1}} \dots \zeta_{i_{l,L_l}} \bar{\zeta}_{j_{l,L_l}}] \bar{\zeta}_{N+1}^s, \\
 &= (1 + |\zeta_{N+1}|^2) \sum_{s \geq 0} \sum_{\sum_{l=1}^{s+1} I_l = s(N+1)+1} \sum_{m_1 = I_1}^N \dots \sum_{m_{s+1} = I_{s+1}}^N \\
 &\quad \sum_1 \dots \sum_{s+1} \prod_{I_l} [\zeta_{m_l} \prod_{k=m_l+1}^N (1 + |\zeta_k|^2) c_{\vec{i}_l - (I_l - 1) \vec{1}, \vec{j}_l - (I_l - 1) \vec{1}} \bar{\zeta}_{i_{l,1}} \bar{\zeta}_{j_{l,1}} \dots \zeta_{i_{l,L_l}} \bar{\zeta}_{j_{l,L_l}}] \bar{\zeta}_{N+1}^s,
 \end{aligned}$$

where for each  $1 \leq l \leq s+1$ , the sum  $\sum_l$  is over multiindices satisfying

$$\begin{array}{ccccccc}
 j_{l,1} & \leq & \dots & \leq & j_{l,L_l} & \leq & N \\
 \vee & & & & \vee & & \\
 m_l & \leq & i_{l,1} & \leq & \dots & \leq & i_{l,L_l}
 \end{array}, \quad \sum_{\tau=1}^{L_l} (j_{l,\tau} - i_{l,\tau}) = m_l - I_l,$$

Consider a term in this sum of the form

$$(5.3) \quad \prod_{I_l} [\zeta_{m_l} \prod_{k=m_l+1}^N (1 + |\zeta_k|^2) \zeta_{i_{l,1}} \bar{\zeta}_{j_{l,1}} \dots \zeta_{i_{l,L_l}} \bar{\zeta}_{j_{l,L_l}}] \bar{\zeta}_{N+1}^s,$$

where  $m_l \leq i_{l,1}$  for each  $l$ . Let  $n = \min\{m_l : 1 \leq l \leq s+1\}$ , and factor out

$$\zeta_n \prod_{k=n+1}^N (1 + |\zeta_k|^2)$$

in (5.3). What remains can be expressed as a positive integral combination of monomials

$$\zeta_{i_1} \bar{\zeta}_{j_1} \zeta_{i_2} \bar{\zeta}_{j_2} \dots \zeta_{i_r} \bar{\zeta}_{j_r},$$

where

$$\begin{array}{ccccccc}
 j_1 & \leq & \dots & \leq & j_L & \leq & N+1 \\
 \vee & & & & \vee & & \\
 n & \leq & i_1 & \leq & \dots & \leq & i_L
 \end{array}, \quad \sum_{l=1}^L (j_l - i_l) = n - 1.$$

Multiplicities arise when the factors with  $m_l \neq m$ ,

$$\prod_{k=m_l+1}^N (1 + |\zeta_k|^2)$$

are expanded. Thus the entire sum can be written as

$$\sum_{n=1}^N \zeta_n \prod_{k=n+1}^{N+1} (1 + |\zeta_k|^2) s_n(\zeta_n, \dots, \zeta_{N+1})$$

with

$$s_n(\zeta_n, \dots, \zeta_{N+1}) = \sum c_{i,j}^{(N+1)} \zeta_{i_1} \bar{\zeta}_{j_1} \zeta_{i_2} \bar{\zeta}_{j_2} \dots \zeta_{i_r} \bar{\zeta}_{j_r},$$

the sum is over multiindices satisfying

$$\begin{array}{ccccccc} j_1 & \leq & \dots & \leq & j_L & \leq & N+1 \\ \vee & & & & \vee & & \\ n & \leq & i_1 & \leq & \dots & \leq & i_L \end{array}, \quad \sum_{l=1}^L (j_l - i_l) = n-1,$$

and  $c_{i,j}^{(N+1)}$  can be computed, in principle, recursively. If  $j_L \leq N$ , then  $c_{i,j}^{(N+1)} = c_{i,j}^{(N)}$ . Otherwise the index  $(i, j)$  has the form

$$\begin{array}{ccccccc} j_1 & \leq & \dots & j_r & < & N+1 & \dots & N+1 \\ \vee & & & \vee & & \vee & & \vee \\ i_0 & \leq & i_1 & \leq & \dots & i_r & \leq & i_{r+1} & \dots & \leq & i_L \end{array}$$

where  $r+s=L$ . The corresponding terms will all originate from the term involving the index  $s$  in the last expression for (5.2). There are many ways that terms could arise, and at best we obtain a formula for  $c^{(N+1)}$  in terms of coefficients  $c^{(N)}$ . So at this point we can only see that these coefficients are positive.  $\square$

Our aim now is to consider another approach which yields a closed formula for "generic"  $c_{i,j}$ . This formula a priori involves signs, and we will make use of Proposition 4 to identify cancellations.

The matrix

$$\begin{pmatrix} 1 & \sum_{j=1}^n x_j^* z^{-j} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 & 0 \\ 0 & a_2^{-1} \end{pmatrix} \begin{pmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & \delta(z) \end{pmatrix} \\ = \begin{pmatrix} a_2 \alpha + x^* a_2^{-1} \gamma & a_2 \beta + x^* a_2^{-1} \delta \\ a_2^{-1} \gamma & a_2^{-1} \delta \end{pmatrix}$$

is special unitary, for all  $z \in S^1$ . Therefore  $-\gamma^* = a_2^2 \beta + x^* \delta$ , and initially assuming  $\delta$  is nonvanishing, this implies  $x^* = P_-(-\gamma^* \delta^{-1})$ . In particular

$$\begin{aligned} x_1^* &= \text{Residue}(-\gamma^* \delta^{-1}) \\ &= -\gamma_1^* + (\gamma_2^* \delta_1 + \gamma_3^* \delta_2 + \dots) - (\gamma_3^* (\delta^2)_2 + \dots) \\ &= -\sum_{m \geq 1} \gamma_m^* \sum (-1)^s \delta_{n_1} \dots \delta_{n_s} \end{aligned}$$

where the second sum is over tuples  $n_1, \dots, n_s \geq 1$  satisfying  $\sum n_l = m-1$ . Using the formulas for  $\gamma^*$  and  $\delta$  in Proposition 2,

$$\begin{aligned} x_1^* &= \sum (-1)^{s+1} ((-1)^{r_m+1} \sum \zeta_{i_{m,1}} \bar{\zeta}_{j_{m,1}} \dots \zeta_{i_{m,r_m}} \bar{\zeta}_{j_{m,r_m}} \zeta_{i_{m,r_m+1}}) \\ &\quad (-1)^{r_{n_1}} (\sum \zeta_{i_{n_1,1}} \bar{\zeta}_{j_{n_1,1}} \dots \zeta_{i_{n_1,r_{n_1}}} \bar{\zeta}_{j_{n_1,r_{n_1}}}) \\ &\quad \dots (-1)^{r_{n_s}} (\sum \zeta_{i_{n_s,1}} \bar{\zeta}_{j_{n_s,1}} \dots \zeta_{i_{n_s,r_{n_s}}} \bar{\zeta}_{j_{n_s,r_{n_s}}}) \end{aligned}$$

where the indexing can be described in the following way: the first sum is over  $m, n_1, \dots, n_s \geq 1$  satisfying  $\sum_l n_l = m-1$ , the first internal sum, or cluster indexed by  $m$ , is over indices satisfying

$$0 < i_{m,1} < j_{m,1} < \dots < j_{m,r} < i_{m,r_m+1}, \quad \sum_{k=1}^{r_m+1} i_{m,k} - \sum_{k=1}^{r_m} j_{m,k} = m$$

and the cluster indexed by  $n_l$  is over indices satisfying

$$0 < i_{n_l,1} < j_{n_l,1} < \dots < j_{n_l,r_{n_l}}, \quad \sum_{k=1}^{r_{n_l}} (j_{n_l,k} - i_{n_l,k}) = n_l.$$

We now write this as a single sum and consider one of the terms. We can put the  $i$ -indices (which are organized in clusters)

$$i_{m,1}, \dots, i_{m,r_m+1}; i_{n_1,1}, \dots, i_{n_1,r_{n_1}}; \dots; i_{n_s,1}, \dots, i_{n_s,r_{n_s}}$$

and the  $j$ -indices

$$j_{m,1}, \dots, j_{m,r_m}; j_{n_1,1}, \dots, j_{n_1,r_{n_1}}; \dots; j_{n_s,1}, \dots, j_{n_s,r_{n_s}}$$

in nondecreasing order, which we write as

$$\mathbf{i}_0 \leq \mathbf{i}_1 \leq \dots \leq \mathbf{i}_L \quad \text{and} \quad \mathbf{j}_1 \leq \dots \leq \mathbf{j}_L,$$

respectively.

**Lemma 3.** *In addition to being nondecreasing, the indices  $\mathbf{i}_l, \mathbf{j}_l$  satisfy  $\mathbf{i}_{l-1} < \mathbf{j}_l$ , for  $l = 1, \dots, L$ .*

*Proof.* With the possible exception of  $i_{m,r+1}$ , for any given  $\mathbf{i}$ -index, it is possible to find a  $\mathbf{j}$ -index with greater value, so that the map from these  $\mathbf{i}$ -indices to  $\mathbf{j}$ -indices is 1-1 (simply map  $i_{n,l}$  to  $j_{n,l}$ ). One of  $\mathbf{i}_{L-1}$  or  $\mathbf{i}_L$  must be strictly less than  $\mathbf{j}_L$ , hence  $\mathbf{i}_{L-1}$  must be strictly less than  $\mathbf{j}_L$ . Similarly one of  $\mathbf{i}_{L-2}$  or  $\mathbf{i}_{L-1}$  or  $\mathbf{i}_L$  must be strictly less than  $\mathbf{j}_{L-1}$ , hence  $\mathbf{i}_{L-2}$  must be strictly less than  $\mathbf{j}_{L-1}$ . Continuing in this way, this implies the strict inequalities in the Lemma.  $\square$

We claim that we can additionally assume that

$$(5.4) \quad \mathbf{i}_l \leq \mathbf{j}_l, \quad l = 1, \dots, L.$$

This is not implied by cluster decomposition considerations. For example the index set

$$\begin{array}{cc} 2 & 2 \\ 1 & 1 \end{array} \quad 3$$

violates (5.4), yet there are two cluster decompositions:  $1 < 2 < 3; 1 < 2$  (with  $(-1)^{s+L} = (-1)^{1+2} = -1$ ) and  $3; 1 < 2; 1 < 2$  (with  $(-1)^{s+L} = (-1)^{2+2} = 1$ ). This claim is justified by Proposition 4, which implies that terms corresponding to indices not satisfying (5.4) will cancel out (It would clearly be desirable to see this cancellation directly, but I do not know how to do this). This implies the following formula.

**Lemma 4.**  $x_1^* = \sum \mathbf{c}_{\mathbf{i},\mathbf{j}} \zeta_{\mathbf{i}_0} \zeta_{\mathbf{i}_1} \bar{\zeta}_{\mathbf{j}_1} \dots \zeta_{\mathbf{i}_L} \bar{\zeta}_{\mathbf{j}_L}$ , where the indices satisfy the constraints

$$(5.5) \quad 0 < \mathbf{i}_0 \leq \mathbf{i}_1 \leq \dots \leq \mathbf{i}_L, \quad \mathbf{j}_1 \leq \dots \leq \mathbf{j}_L, \quad \mathbf{i}_1 \leq \mathbf{j}_1, \dots, \mathbf{i}_L \leq \mathbf{j}_L,$$

$$\mathbf{i}_0 < \mathbf{j}_1, \dots, \mathbf{i}_{L-1} < \mathbf{j}_L, \quad \sum \mathbf{i} - \sum \mathbf{j} = 1,$$

and

$$(5.6) \quad \mathbf{c}_{\mathbf{i},\mathbf{j}} = \sum (-1)^{s+L},$$

where the sum is over all possible ways in which the indices can be partitioned as

$$\begin{aligned} & i_{m,1}, \dots, i_{m,r_m+1}; i_{n_1,1}, \dots, i_{n_1,r_{n_1}}; \dots; i_{n_s,1}, \dots, i_{n_s,r_{n_s}} \\ & j_{m,1}, \dots, j_{m,r_m}; j_{n_1,1}, \dots, j_{n_1,r_{n_1}}; \dots; j_{n_s,1}, \dots, j_{n_s,r_{n_s}} \end{aligned}$$

so that the strict interlacing inequalities

$$0 < i_{m,1} < j_{m,1} < \dots < j_{m,r} < i_{m,r+1}, \quad \sum_k i_{m,k} - \sum_k j_{m,k} = m$$

and

$$0 < i_{n_l,1} < j_{n_l,1} < \dots < j_{n_l,r}, \quad \sum_k (j_{n_l,k} - i_{n_l,k}) = n_l$$

hold for  $l = 1, \dots, s$ .

To compare with the formula in Proposition 4, we first sum over  $n = \mathbf{i}_0$ , and write

$$(5.7) \quad x_1^* = \sum_{n=1}^{\infty} \zeta_n \sum \mathbf{c}_{(n,\mathbf{i}),\mathbf{j}} \zeta_{\mathbf{i}_1} \bar{\zeta}_{\mathbf{j}_1} \dots \zeta_{\mathbf{i}_L} \bar{\zeta}_{\mathbf{j}_L}$$

where  $(n, \mathbf{i})$  now stands for  $n \leq \mathbf{i}_1 \leq \dots \leq \mathbf{i}_L$ . This implies

$$(5.8) \quad \sum \mathbf{c}_{(n,\mathbf{i}),\mathbf{j}} \zeta_{\mathbf{i}_1} \bar{\zeta}_{\mathbf{j}_1} \dots \zeta_{\mathbf{i}_L} \bar{\zeta}_{\mathbf{j}_L} = \left( \prod_{k=n+1}^{\infty} (1 + |\zeta_k|^2) \right) \sum c_{i,j} \zeta_{i_1} \bar{\zeta}_{j_1} \zeta_{i_2} \bar{\zeta}_{j_2} \dots \zeta_{i_r} \bar{\zeta}_{j_r}$$

where the indexing set for the latter sum satisfies the constraints in Proposition 4. To directly compare the coefficients we expand the product of factors  $(1 + |\zeta_j|^2)$  and distribute the pairs  $\zeta_j$  and  $\bar{\zeta}_j$ . This implies the following

**Lemma 5.** *Consider an index as in (5.5), with  $n = \mathbf{i}_0$ .*

- (a) *If  $\{\mathbf{i}_l\} \cap \{\mathbf{j}_{l'}\}$  is null, then  $\mathbf{c}_{(n,\mathbf{i}),\mathbf{j}} = c_{\mathbf{i},\mathbf{j}}$ .*
- (b) *In general*

$$\mathbf{c}_{(n,\mathbf{i}),\mathbf{j}} = \sum c_{i,j},$$

where the sum is over all subindexing sets of  $(n, \mathbf{i}, \mathbf{j})$ , resulting from cancellation of pairs  $\mathbf{i}_l = \mathbf{j}_{l'}$ , which satisfy the constraints in Proposition 4.

- (c) *In particular for any indexing set  $(i, j)$  as in Proposition 4,  $c_{i,j} \leq \mathbf{c}_{(n,\mathbf{i}),j}$ .*

**Example.** *To clarify (b), given an indexing set such as*

$$\begin{array}{ccc} 5 & 6 & 7 \\ 3 & 4 & 5 & 6 \end{array}$$

there are three proper subindexing sets,

$$\begin{array}{ccc} 6 & 7 & & 5 & 7 & & 7 \\ 3 & 4 & 6 & 3 & 4 & 5 & 3 & 4 \end{array}$$

Part (a) of Lemma 5, and Lemma 4, yield an expression for a generic  $c_{i,j}$ , where generic is defined by the null intersection condition in (a). Using this formula it is possible to write “most” of the terms in  $s_{n,r}$  in Proposition 4 in terms of products of the Hermitian expressions

$$b_n(m) = \zeta_n \bar{\zeta}_{n+m} + \zeta_{n+1} \bar{\zeta}_{n+1+m} + \dots$$

These expressions can be estimated using Cauchy-Schwarz, and they are also easy to understand in probabilistic contexts. Unfortunately I do not know how to systematically estimate nongeneric terms.



**Example.**

$$s_2 = s_{2,1} = b_2(1) + b_3(1)$$

and in general

$$s_{n,1} = b_n(n-1) + b_{n+1}(n-1)$$

$s_{3,2}$  is a quadratic expression in terms of the variables  $\zeta_3\bar{\zeta}_4, \zeta_4\bar{\zeta}_5, \dots$ . The matrix is

$$\begin{array}{cccccc} 1 & 3 & 2 & 2 & 2 & .. \\ & 3 & 6 & 4 & 4 & 4 & .. \\ & & 3 & 6 & 4 & 4 & 4 & .. \\ & & & 3 & 6 & 4 & 4 & 4 & .. \end{array}$$

Therefore

$$s_{3,2} = b_3(1)^2 + b_4(1)^2 + \sum_{i \geq 4} \zeta_i \bar{\zeta}_{i+1} \zeta_i \bar{\zeta}_{i+1} + \zeta_3 \bar{\zeta}_4 \zeta_4 \bar{\zeta}_5 + 2 \sum_{i \geq 4} \zeta_i \bar{\zeta}_{i+1} \zeta_{i+1} \bar{\zeta}_{i+2}$$

Thus “most” of  $s_{3,2}$  can be written in terms of powers of Hermitian expressions, and two “diagonal” sums near the boundary of the cone that we are adding over.

**5.2.  $\zeta$  in terms of  $x$ .** We have  $\zeta_n = \zeta_1(x_n, x_{n+1}, \dots)$ , and for a finite number of variables, one can generate formulas for  $\zeta_1$ . For example, if  $p_n = \prod_{j > n} (1 + |\zeta_j|^2)$ , then

$$\begin{aligned} \zeta_1(x_1, x_2, x_3, x_4) = & \frac{1}{p_1} x_1 - \frac{1}{p_1 p_2 p_3} x_2^2 \bar{x}_3 + 2 \frac{1}{p_1 p_2 p_3^2 p_4} x_2 x_3^2 \bar{x}_3 \bar{x}_4 - 2 \frac{1}{p_1 p_3 p_4} x_2 x_3 \bar{x}_4 \\ & - \frac{1}{p_1 p_2 p_3^3 p_4^2} x_3^4 \bar{x}_3 \bar{x}_4^2 + \frac{1}{p_1 p_3^2 p_4^2} x_3^3 \bar{x}_4^2, \end{aligned}$$

where the  $p_i$  can be expressed in terms of  $x$  using the displayed line following (6.10) in [4]. But I have not made any progress toward finding a general formula.

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